

UNIVERSALITY FOR SU(2) YANG-MILLS THEORY IN (2+1)D

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Abstract

The Green's Function Monte Carlo method of Chin et al is applied to SU(2) Yang-Mills theory in (2+1)D. Accurate measurements are obtained for the ground-state energy and mean plaquette value, and for various Wilson loops. The results are compared with series expansions and coupled cluster estimates, and with the Euclidean Monte Carlo results of Teper. A striking demonstration of universality between the Hamiltonian and Euclidean formulations is obtained.

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I. INTRODUCTION

The Euclidean Monte Carlo approach is now well entrenched as the most accurate and reliable method of studying lattice gauge theory in the weak-coupling regime. In the Hamiltonian formulation, however, the development of Monte Carlo methods has been less rapid, and much work remains to be done.

Most of the early attempts at Monte Carlo calculations in the Hamiltonian formulation [1–5] were studies of U(1) Yang-Mills theory, using a strong-coupling (electric field) representation, which gives rise to a discrete set of basis states. Unfortunately, such techniques appear to fail for non-Abelian theories, because the introduction of Clebsch-Gordan coefficients leads to the infamous “minus sign” problem. That is, the Hamiltonian in this representation gives rise to amplitudes of both positive and negative signs contributing to the same final state, which interfere with each other. Monte Carlo techniques are not able to recognize and cancel these interfering amplitudes, and consequently they build up as “noise” in the calculation. One needs to develop methods based on a weak-coupling representation, which should be able to avoid this problem.

A weak-coupling algorithm has in fact been introduced by Heys and Stump [6] and Chin, Negele and Koonin [7], based on the Green’s Function Monte Carlo (GFMC) techniques of Kalos and collaborators [8,9]. We have previously used an improved version of this technique to perform an accurate finite-size scaling study of the U(1) gauge model in (2+1) dimensions [10]. Here we report on an application of the same technique to a non-Abelian model, SU(2) Yang-Mills theory in (2+1)D.

A theoretical discussion of the SU(2) model has been given by Feynman [11], who argued that any correlations would be of finite range, and that the flux between external charges is restricted to a “tube” of finite extent, leading to a strongly confining linear potential. All physical quantities should simply scale according to their physical dimensions in the continuum limit, so that the mass gap behaves as

$$Ma \sim c_1 g^2 \quad \text{as} \quad a \rightarrow 0 \quad (1.1)$$

and the string tension behaves as

$$\sigma a^2 \sim c_2 g^4 \quad \text{as} \quad a \rightarrow 0 \quad (1.2)$$

where a is the lattice spacing, and $g^2 = e^2 a$ is the dimensionless coupling.

Numerical treatments have borne out these expectations very well, including both variational [12–15] and numerical [16–22] calculations. In the Hamiltonian formulation, most studies have employed some form of linked cluster expansion, such as strong-coupling series expansions [21,22], coupled-cluster expansions [23,24], and the so-called [21] “Exact Linked-Cluster Expansion” (ELCE). One of our major aims is to make a comparison between the results of the GFMC method and these linked-cluster expansion methods.

In the Euclidean formulation, Teper [25] has recently applied the full power of modern Monte Carlo techniques to this model. His results are very accurate, and now provide the benchmark with which all other approaches must be compared at weak coupling.

In Section II and III of the paper we briefly summarize the Monte Carlo method and the coupled-cluster method to be employed, and in Section IV we present the results. The

GFMC results are extremely good for the ground-state energy and mean plaquette value, but not as good as the linked-cluster estimates for excited-state properties such as the string tension. A comparison with Teper's results [25] gives striking evidence of universality between the Hamiltonian and Euclidean formulations, not only in the continuum limit, but even at finite couplings. Further discussion is given in Section V.

II. MONTE CARLO METHOD

We have employed a Green's Function Monte Carlo approach to SU(2) Hamiltonian lattice gauge theory which was discussed previously by Chin et al [26], and we shall merely summarize some of the key points here. The lattice Hamiltonian is given by

$$H = \frac{g^2}{2a} \left\{ \sum_l E_l^a E_l^a - \lambda \sum_p \text{Tr} U_p \right\} \quad (2.1)$$

where E_l^a is a component of the electric field at link l , $\lambda = 4/g^4$, and U_p denotes the product of four link operators U_l around an elementary plaquette. The commutation relation between electric field and link operators at each link may be taken as:

$$[E_l^a, U_l] = \frac{1}{2} \tau^a U_l, \quad (2.2)$$

choosing the E_l^a as left generators of SU(2). For calculational purposes, it is most convenient to "scale out" a factor g^2/a , and work with the dimensionless Hamiltonian

$$H = \frac{1}{2} \sum_l E_l^a E_l^a - \frac{\lambda}{2} \sum_p \text{Tr} U_p. \quad (2.3)$$

This will be assumed henceforth, unless stated otherwise.

The Green's Function Monte Carlo technique employs the operator $\exp(-\tau(H - E))$ as a projector onto the ground state $|\Psi_0\rangle$:

$$|\Psi_0\rangle \propto \lim_{\tau \rightarrow \infty} e^{-\tau(H-E)} |\Phi\rangle \quad (2.4)$$

where $|\Phi\rangle$ is any suitable trial state. To procure some variational guidance, one performs a "similarity transformation" with the trial wave function Φ , and evolves the product $\Phi|\Psi_0\rangle$ in imaginary time. The heart of the procedure is the calculation of the matrix element corresponding to a single small time step $\Delta\tau$: Chin et al [26] show that

$$\begin{aligned} & \langle \mathbf{x}' | \Phi e^{-\Delta\tau(H-E)} \Phi^{-1} | \mathbf{x} \rangle \\ &= \prod_l \langle U_l' | N \{ \exp(-\frac{1}{2} \Delta\tau E_l^a E_l^a) \exp[\Delta\tau E_l^a (E_l^a \ln \Phi)] \} | U_l \rangle \\ & \quad \exp\{\Delta\tau [E - \Phi^{-1} H \Phi(x)]\} + O(\Delta\tau^2) \\ & \equiv p(\mathbf{x}', \mathbf{x}) w(x) + O(\Delta\tau^2) \end{aligned} \quad (2.5)$$

where $\mathbf{x} = \{U_l\}$ denotes an entire lattice configuration of link fields.

In the Monte Carlo procedure, the product $\Phi|\Psi\rangle$ is simulated by the density of an ensemble of random walkers in configuration space. At the k th time step, the “weight” of each walker at \mathbf{x}_k^i is multiplied by $w(\mathbf{x}_k^i)$, and the next ensemble $\{\mathbf{x}_{k+1}^i\}$ is evolved from $\{\mathbf{x}_k^i\}$ according to the matrix element $p(\mathbf{x}_{k+1}, \mathbf{x}_k)$. Chin *et al.* [26] show that the effect of $p(\mathbf{x}_{k+1}, \mathbf{x}_k)$ is to alter each link variable U in $\{\mathbf{x}_k\}$ to U' by a Gaussian random walk plus a “drift step” guided by the trial function:

$$U' = \Delta U U_d U \quad (2.6)$$

where $U_d = \exp[i\frac{1}{2}\tau^a(i\Delta\tau E^a \ln \Phi)]$ is the drift step, and ΔU is an $SU(2)$ group element Gaussian distributed in distance from the identity with zero mean and variance $\langle \Delta s^2 \rangle = 3\Delta t$.

At the end of each iteration or “time step” $\Delta\tau$, the trial energy E is adjusted to compensate for any change in the total weight of all walkers in the ensemble; so that at equilibrium its average value is equal to the ground-state energy E_0 . A “branching” process is also carried out: any walker whose weight has gone above (say) 2 is split into two new walkers, while any two walkers with weights less than (say) 1/2 are combined into one, chosen randomly according to weight from the originals. This procedure of “Runge smoothing” [27] maximizes statistical accuracy by keeping the weights of all the walkers within fixed bounds, while minimizing any fluctuations in the total weight due to the branching process. Various corrections due to the finite time interval $\Delta\tau$ have been ignored in this discussion, and the limit $\Delta\tau \rightarrow 0$ must be taken in some fashion to eliminate such corrections.

The link variables are elements of the group $SU(2)$, and are most conveniently represented by quaternions $x^\mu = (x^0, x^a)$

$$U = \exp(-i\frac{1}{2}\tau^a A^a) = \cos(\frac{1}{2}\rho) - i\tau^a \hat{n}^a \sin(\frac{1}{2}\rho) \equiv x^0 - i\tau^a x^a \quad (2.7)$$

where $\{a = 1, 2, 3\}$, $\rho^2 = A^a A^a$, $\hat{n}^a = A^a/\rho$, and

$$x^0 x^0 + x^a x^a = 1 \quad (2.8)$$

Note that then

$$\text{Tr}(U) = 2x^0. \quad (2.9)$$

The product of two link variables is then easily found by quaternion multiplication:

$$U(x)U(y) = (x^0 y^0 - x^a y^a) - i\tau^a (x^0 y^a + y^0 x^a + \epsilon^{abc} x^b y^c) \quad (2.10)$$

To generate the Gaussian-distributed element ΔU near the identity, we [26] simply generate 3 random numbers A^1, A^2, A^3 from a Gaussian distribution with zero mean and variance $\Delta\tau$, and set

$$x^\mu = (\cos(\frac{1}{2}\rho), \frac{A^a}{\rho} \sin(\frac{1}{2}\rho)) \quad (2.11)$$

where $\rho^2 = A^a A^a$.

The trial wave function is chosen to be the one-parameter form [26]:

$$\Phi = \prod_p \exp[\frac{\alpha}{2} \text{Tr} U_p] \quad (2.12)$$

Then the drift step is

$$\begin{aligned} U_d &= \exp[i\frac{1}{2}\tau^a(i\Delta\tau E^a \ln \Phi)] \\ &= \exp(-i\frac{1}{2}\tau^a A_l^a) \end{aligned} \quad (2.13)$$

where

$$A_l^a = -\frac{1}{2}\alpha\Delta\tau \sum_{p \in l} x_{p,l}^a \quad (2.14)$$

and the sum runs over plaquettes adjacent to the link l , with the $x_{p,l}^a$ being “rotated” quaternion elements as defined in Appendix A of Chin *et al.* [26]. Finally, the trial function energy factor is

$$\Phi^{-1} H \Phi = \sum_l \sum_{p \in l} \left\{ -\frac{\alpha^2}{8} x_{p,l}^a x_{p,l}^a + \frac{1}{4} (3\alpha/2 - \lambda) x_p^0 \right\} \quad (2.15)$$

where x_p^0 is the zeroth quaternion element of the plaquette operator or group element U_p , and the sum $p \in l$ runs over the two plaquettes adjacent to the link l .

The expectation value of an observable Q in the ground state can be measured by the technique of Hamer *et al.* [10]. If one adds a small perturbation yQ to the Hamiltonian

$$H' = H + yQ \quad (2.16)$$

then by the Feynman-Hellmann theorem the required expectation value is given by

$$\langle Q \rangle_0 = \left. \frac{dE'_0}{dy} \right|_{y=0} \quad (2.17)$$

Perform a Taylor expansion in y for the eigenvector and eigenvalue

$$|\Psi_0(y, \tau)\rangle = |\Psi_0^0(\tau)\rangle + y|\Psi_0^1(\tau)\rangle + O(y^2) \quad (2.18)$$

$$E'_0(y) = E_0^0 + yE_0^1 + O(y^2) \quad (2.19)$$

substitute in the evolution equation (2.4) and equate powers of y to obtain:

$$|\Psi_0^0(\tau + \Delta\tau)\rangle = e^{-\Delta\tau(H-E)} |\Psi_0^0(\tau)\rangle \quad (2.20)$$

(which is the same as in the unperturbed case), and

$$|\Psi_0^1(\tau + \Delta\tau)\rangle = e^{-\Delta\tau(H-E)} |\Psi_0^1(\tau)\rangle + \Delta\tau(E_0^1 - Q) |\Psi_0^0(\tau)\rangle \quad (2.21)$$

Equation (2.21) is an evolution equation for $|\Psi_0^1\rangle$ of similar structure to (2.20). It is simulated by giving a “secondary” weight to each walker in the ensemble to simulate $|\Psi_0^1\rangle$, and evolving it according to (2.21). The value of E_0^1 is adjusted to keep the total of all secondary weights constant after each iteration. At equilibrium, its average value gives an estimate of $\langle Q \rangle_0$.

III. COUPLED CLUSTER EXPANSION METHOD

The coupled cluster expansion method has been extensively used in many-body theory, and has recently been introduced to lattice gauge theory by Bishop [28], Llewellyn-Smith and Watson [24], and Guo *et al* [23], although the truncation schemes used there are different. The basic idea of this expansion is to assume the ground state $|\Psi_0\rangle$ and first excited state (the glueball wavefunction) $|\Psi\rangle$ of the Hamiltonian in eq. (2.1) can be represented by an exponential form:

$$|\Psi_0\rangle = e^{R(U)}|0\rangle \quad (3.1)$$

$$|\Psi\rangle = F(U)e^{R(U)}|0\rangle$$

where $R(U)$ and $F(U)$ are functions of loop variables and the state $|0\rangle$ is the strong-coupling ground-state, defined by

$$E_l^a|0\rangle = 0. \quad (3.2)$$

The eigenvalue equation for H can then be written as:

$$\sum_l ([E_l, [E_l, R]] + [E_l, R][E_l, R]) - \frac{4}{g^4} \sum_p \text{Tr}(U_p) = \frac{2a}{g^2} \epsilon_0 \quad (3.3)$$

$$\sum_l ([E_l, [E_l, F]] + 2[E_l, F][E_l, R]) = \frac{2a}{g^2} (\epsilon_1 - \epsilon_0)$$

where ϵ_0 (ϵ_1) is the ground state (the first excited state) energy. $R(U)$ and $F(U)$ can be decomposed according to the order of graphs,

$$R = \sum_i R_i \quad (3.4)$$

$$F = \sum_i F_i$$

and the lowest order term of R and F is:

$$R_1 = c_1 \square \quad (3.5)$$

$$F_1 = f_1 \square$$

The graphs of order i are generated by

$$\sum_{j=1}^{i-1} [E_l, R_j][E_l, R_{i-j}] \quad (3.6)$$

in equation (3.3). In order to make the calculation possible, some truncation scheme to truncate the eigenvalue equation must be used. The truncation scheme used by Llewellyn-Smith and Watson [24] is

$$\sum_l ([E_l, [E_l, \sum_{i=1}^n R_i]] + \sum_{i,j=1}^n [E_l, R_i][E_l, R_j]) - \frac{4}{g^4} \sum_p \text{Tr}(U_p) = \frac{2a}{g^2} \epsilon_0 \quad (3.7)$$

$$\sum_l ([E_l, [E_l, \sum_{i=1}^n F_i]] + 2 \sum_{i,j=1}^n [E_l, F_i][E_l, R_j]) = \frac{2a}{g^2} (\epsilon_1 - \epsilon_0)$$

where the new graphs generated by $[E_l, R_i][E_l, R_j]$ and $[E_l, F_i][E_l, R_j]$ are simply discarded. Guo *et al* [23] have argued that because of this, the continuum limit of this system could not be preserved, and they have proposed a better truncation scheme:

$$\sum_l ([E_l, [E_l, \sum_{i=1}^n R_i]] + \sum_{i+j \leq n} [E_l, R_i][E_l, R_j]) - \frac{4}{g^4} \sum_p \text{Tr}(U_p) = \frac{2a}{g^2} \epsilon_0 \quad (3.8)$$

$$\sum_l ([E_l, [E_l, \sum_{i=1}^n F_i]] + 2 \sum_{i+j \leq n} [E_l, F_i][E_l, R_j]) = \frac{2a}{g^2} (\epsilon_1 - \epsilon_0)$$

The most tedious task for the high-order approximations is to generate a list of independent loop configurations and derive the nonlinear coupled equations. So far, all these calculations in lattice gauge theory have been carried out by hand. We have tried to develop computer algorithms to overcome this problem. Borrowing some ideas from our computer algorithms used to generate a list of clusters for our linked-cluster series expansions and t -expansions [30], a preliminary program was developed. Up to fourth order, a list of 70 graphs was generated, whereas Llewellyn-Smith and Watson [24] only obtained 69 graphs by hand. Some results from the truncation scheme (3.8) were presented in a previous paper [29]. Here we make a comparison of the results of different truncation schemes with the GFMC simulation.

IV. RESULTS

The GFMC method was used to calculate the ground-state energy, mean plaquette value, and some Wilson loop expectation values for lattice sizes of 2×2 up to 6×6 sites, all with periodic boundary conditions. At each coupling λ , the variational parameter α was adjusted to minimize the error by a series of trial runs. Production runs typically employed an ensemble size of 2000 walkers for 50,000 iterations, with step sizes $\Delta\tau = 0.05$ and 0.01 , followed by a linear extrapolation to $\Delta\tau = 0$. The first 8K iterations were discarded to allow for equilibration, and the results were averaged over blocks of 5K iterations before estimating the error to minimize correlation effects.

Ground-State Energy

A table of estimates of the ground-state energy per site is given in Table I. The finite-size dependence of these results is illustrated in Figure 1. It can be seen that in the strong-coupling region (small λ) the convergence to the bulk limit is very rapid. At weak coupling

(large λ), however, the convergence is slower, as one might expect, and at $\lambda = 16$ the data can be quite well fitted by a straight line in $1/L^3$ (where L is the lattice size). The straight line fit in Fig. 1c corresponds to

$$\omega_0(L) = -10.81 - 1.19/L^3 \quad (4.1)$$

This behaviour appears similar to that of the U(1) model [10], and we shall discuss it further in the concluding section.

It can be seen from Table I that the estimated bulk limits are in excellent agreement with earlier strong-coupling series estimates [22], within errors, which provides a check that our algorithm is working correctly. It is also noteworthy that in the weak-coupling region beyond $\lambda = 4$ the Monte Carlo results are much more accurate than the series estimates.

The data in the weak-coupling region can be compared with the asymptotic weak-coupling expansion obtained by Hofsäss and Horsley [14]:

$$\omega_0 \sim -\lambda + 1.4372\lambda^{1/2} \quad \text{as } \lambda \rightarrow \infty \quad (4.2)$$

Figure 2 shows that the Monte Carlo results appear to be in excellent agreement with this prediction as the continuum limit is approached. The fit to the data shown in Figure 2 corresponds to

$$\omega_0 = -\lambda + 1.4372\lambda^{1/2} - 0.59(2) \quad (4.3)$$

Mean Plaquette Value

The mean plaquette value

$$P = \frac{1}{2} \langle \text{Tr} U_p \rangle_0 = -\frac{1}{N} \frac{d\omega_0}{d\lambda} \quad (4.4)$$

by the Feynman-Hellmann theorem. This expectation value has also been estimated by the methods outlined in the previous Section. The results are listed in Table 2: again, it can be seen that the Monte Carlo estimates of the bulk limit are in excellent agreement with previous strong-coupling series estimates [22]. The Monte Carlo results are again more accurate than the series estimates beyond $\lambda = 4$.

The behaviour in the weak-coupling region is illustrated in Figure 3. The data are described almost exactly by the asymptotic prediction obtained from (4.2):

$$1 - P \sim 0.7186\lambda^{-1/2} \quad \text{as } \lambda \rightarrow \infty \quad (4.5)$$

String Tension

In attempting to estimate the string tension, we have used the method of Wilson loops. The Wilson loop expectation values, defined as

$$W(I, J) = \frac{1}{2} \langle \text{Tr} U(I, J) \rangle_0 \quad (4.6)$$

where $U(I, J)$ denotes a product of link variables around a rectangular loop of $I \times J$ links, were computed using the method of “secondary weights” outlined in the previous section. These can be combined to form the Creutz ratios:

$$\chi(I, J) = -\ln \left[\frac{W(I, J)W(I-1, J-1)}{W(I, J-1)W(I-1, J)} \right] \rightarrow a^2 \sigma \quad \text{as } I, J \rightarrow \infty \quad (4.7)$$

where σ is the string tension. Table 2 shows some typical Wilson loop estimates. Some problems are immediately apparent. At small λ , the estimates decrease extremely rapidly with the size of the loop; while at large λ , they decrease very slowly, and larger lattices are really needed to get reliable values for the string tension. In both cases, the resulting error in the string tension estimate is very large.

Figure 4 shows the GFMC estimates of the string tension, obtained from $\chi(2, 2)$ as measured on the 6×6 lattice, together with some earlier Hamiltonian estimates by Hamer and Irving [21] obtained using the ELCE method, and the Euclidean results of Teper [25]. A number of points may be noted:

i) The Euclidean MC results are an order of magnitude more accurate than any Hamiltonian estimates to date, and extend to weaker couplings;

ii) The GFMC results are relatively inaccurate and appear to diverge at weak couplings. We take this to indicate that both the lattice size and the size of the Wilson loops are too small. Indeed, the $\chi(3, 3)$ estimates appear somewhat lower than $\chi(2, 2)$, but are so inaccurate as to be not worth plotting. The GFMC and ELCE results do not agree; but this is expected, because the first is a ‘spacelike’ (correlation length) estimate, while the second is a ‘timelike’ (energy) estimate, and the two differ by a factor equal to the ‘speed of light’.

iii) On the positive side, it can be seen that the ELCE estimates for the string tension approach a limit very similar to Teper’s results [25] as $g \rightarrow 0$. This provides nice evidence of universality between the Euclidean and Hamiltonian formulations in the continuum limit.

Let us expand on the last point a little further. When g is non-zero, the Euclidean and Hamiltonian results are not directly comparable because there is a difference in scale (i.e. coupling) between the two formulations, and the speed of light in the Hamiltonian is not equal to unity. In four dimensions, the relationship between the two scales was calculated long ago to one-loop order in weak-coupling perturbation theory by Hasenfratz and Hasenfratz [31]. The calculation has recently been repeated for the three-dimensional case by one of us [32]. The results for this model may be summarized as:

$$\frac{1}{g_H^2} = \frac{1}{g_E^2} - 0.01924 + O(g_E^2) \quad (4.8)$$

where g_H , g_E are the couplings in the Hamiltonian and Euclidean models, respectively; and

$$c = 1 - 0.08365g_E^2 + O(g_E^4) \quad (4.9)$$

for the ‘speed of light’, with the Hamiltonian normalized as in equation (2.1). To make a direct comparison between string tension estimates, the ‘timelike’ ELCE estimates must be divided by a factor of c , and shifted from coupling g_H to the equivalent g_E , given by (4.8). The revised estimates are then given by

$$\frac{4a\sigma_E^{1/2}}{g_E^2} = \frac{4a\sigma_H^{1/2}}{g_H^2} [1 + 0.0616g_E^2] \quad (4.10)$$

and are compared with Teper's results in Figure 5. It can be seen that the two sets of estimates now lie almost on the same curve, and the remaining discrepancy (of order 3%) may easily be attributed to higher-order (two loop) corrections. This provides even stronger evidence of universality between the two formulations.

Mass Gap

It is much more difficult to measure the energy of an excited state than it is for the ground state using the GFMC method [10], and we have no GFMC results to report for the mass gap. There have been a number of Hamiltonian estimates of the mass gap using various linked cluster expansion techniques [23,24], however, and it is interesting to compare these with the Euclidean results of Teper [25].

Let us begin with estimates of the dimensionless ratio $M/\sqrt{\sigma}$. Figure 6 compares the results obtained by Hamer and Irving [21] with the Euclidean results of Teper [25]. The Hamiltonian estimates have been slightly shifted from coupling g_H^2 to the equivalent coupling g_E^2 using equation (4.8), and 'renormalized' by the speed of light c according to

$$\frac{M_E}{\sqrt{\sigma_E}} = \frac{M_H}{\sqrt{\sigma_H c}} = \frac{M_H}{\sqrt{\sigma_H}} (1 + 0.0418 g_E^2) \quad (4.11)$$

It can be seen that the two sets of data match rather well: once more, in excellent agreement with universality.

Next, Figure 7 shows a number of different Hamiltonian estimates of the mass gap itself, or more precisely of the quantity Ma/g^2 , which should approach a finite value in the continuum limit. The strong-coupling series expansion estimates [22] rise fairly abruptly around $\beta = 4/g^2 \simeq 3$, and then begin to level off towards an asymptotic value estimated previously [22] as 2.22(5). The higher-order coupled-cluster estimates by the truncation schemes used by Llewellyn-Smith and Watson [24] and Guo *et al.* [23] also show a rise until $\beta \simeq 4$, but their behaviour beyond that point is somewhat variable. Teper [25] does not give direct estimates of the mass gap itself; but using his results for σ and the ratio $M/\sqrt{\sigma}$, together with equations (4.8) and (4.9), one may infer a behaviour for the Hamiltonian mass gap at weak coupling which is shown as a heavy solid line in Fig. 7. It would appear that Ma/g^2 actually reaches a peak at around $\beta \simeq 4$, and then declines slowly towards the asymptotic value 1.59(2). The strong-coupling series extrapolations were unable to pick up this decrease at weak coupling. Some of the coupled-cluster approximants perhaps give some indication of it, but not in a very consistent or reliable manner.

V. DISCUSSION

The most interesting result of this paper is the remarkable demonstration of universality between the Euclidean and Hamiltonian formulations which has been obtained. If account is taken of the difference in scale between the two formulations as predicted in one-loop perturbation theory [32], then the Euclidean results of Teper [25] and the Hamiltonian results of Hamer and Irving [21] for the string tension and the ratio $M/\sqrt{\sigma}$ fall almost on top of each other, not only in the continuum limit, but over a whole range of weak couplings. This provides pleasing confirmation of the hypothesis of universality. There is little doubt

that the hypothesis is correct, but it is important to check it wherever possible, since it underpins the whole program of lattice gauge theory.

One of our objectives has been to continue the development of Monte Carlo techniques for Hamiltonian lattice gauge theory, which currently lag behind Euclidean techniques by about a decade. We have shown that the GFMC method of Chin *et al.* [7,26], when improved by “Runge smoothing” [27], is capable of giving an accurate description of the ground-state of this non-Abelian model deep into the weak-coupling regime. The ground-state energy has been obtained to an accuracy of around 2 parts in 10^4 , and the mean plaquette to around 0.3%. The results agree very nicely with both weak-coupling and strong-coupling expansions for the model.

An interesting question concerns the finite-size scaling behaviour to be expected in this model. For the U(1) model, an argument based on effective Lagrangian theory predicts [33] that the ground-state energy per site scales as

$$\omega_0(L) - \omega_0(\infty) \sim -\frac{0.7188c}{L^3} \quad (5.1)$$

where c is the ‘speed of light’, based on the presence of one massless photon degree of freedom per site.¹ This has been confirmed by numerical calculations [10]. The SU(2) model has three gauge boson degrees of freedom per site, and so similar arguments would predict for this model

$$\omega_0(L) - \omega_0(\infty) \sim -\frac{2.1564c}{L^3} \quad (5.2)$$

or, with the normalization of eq. (2.3):

$$\omega_0(L) - \omega_0(\infty) \sim -\frac{4.14}{L^3} \quad (5.3)$$

This does not agree at all well with the numerical results, eq. (4.1). Now in fact the mass gap does not vanish in this model, but remains finite in the continuum limit, so it is not unexpected that the “zeroth order” prediction (5.3) should break down. It would be interesting to carry out a higher-order calculation in weak-coupling perturbation theory for this model to explore these questions further. Such a calculation was carried out for the case of free boundary conditions by Müller and Rühl [34], who obtained a prediction for the mass gap which was an order of magnitude smaller than the numerical results.

For excited state properties, the GFMC techniques have not been so successful as yet [10]. An attempt was made to estimate the string tension by means of expectation values for Wilson loops. The results were not very accurate, and it was clear that the lattice sizes used were too small at weak coupling. The calculation of each Wilson loop involved the carriage of a corresponding “secondary weight” for each walker in the ensemble, which is rather laborious and expensive in computer time. It seems clear that new methods are needed here. One possibility which needs exploration is to see if correlation functions can

¹There is a delicate question of scale involved here, for which see refs. [33,10].

be estimated by averaging over configurations in the ensemble in some fashion, as is done in the Euclidean framework.

A second major objective was to compare the results of various Hamiltonian linked-cluster expansions for this model [21–24] with the Monte Carlo results. It was found that the GFMC results are already superior to linked-cluster estimates for the ground-state energy and mean plaquette at weak coupling, but are not yet competitive for excited state properties, as discussed above.

In the case of the mass gap itself, it was possible to compare the different linked-cluster expansion techniques [22–24]. Out to $\beta \simeq 4$, they were found to agree with each other quite well. Beyond that point, the convergence is poor. Teper’s data [25] imply a gradual decline in the scaled mass gap aM/g^2 to an asymptotic limit of 1.59(2), which is not clearly indicated by any of the expansion methods. It appears, once again, that one must be cautious about placing too much trust in extrapolated linked-cluster expansions deep in the weak-coupling regime. This is not surprising, since the cluster expansions are basically strong-coupling expansions of one or another form, and they cannot be expected to converge very well in the neighbourhood of the weak-coupling (continuum) limit, which is likely to be an essential singularity.

The linked-cluster expansion techniques have an important role to play, nevertheless. They are presently more accurate than the GFMC method for excited states, as outlined above, and will remain useful as a check on the Monte Carlo results at strong and intermediate couplings. They can also be used for models containing dynamical fermions, for instance, which at present are inaccessible by the GFMC method.

At present we are trying to improve the coupled cluster expansion technique [36] by combining it with the D-function expansion used in the “ELCE” approach of Irving, Preece and Hamer [35]. This avoids both the use of the Cayley Hamilton relation for the elimination of redundancies, and also the explicit handling of many $SU(n)$ coupling coefficients, and allows one to define the truncation with respect to an orthogonal basis. The incorporation of states in the spectrum having arbitrary lattice momentum and lattice angular momentum has also been possible in this framework [36].

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FIGURES

FIG. 1. Finite-size dependence of the ground-state energy per site $\omega_0(L)$ at couplings $\lambda = 4, 8$ and 16. The curves in a) and b) are merely to guide the eye; the curve in c) is a straight line.

FIG. 2. Graph showing the weak-coupling behaviour of the ground-state energy per site ω_0 . The straight line corresponds to the asymptotic weak-coupling prediction [15], equation (4.2)

FIG. 3. Weak-coupling behaviour of the mean plaquette value P . The points are the Monte Carlo data, while the dashed line corresponds to the asymptotic weak-coupling prediction (4.5).

FIG. 4. The string tension σ at weak coupling. Circles: Euclidean MC data [25]; squares: ELCE data [21]; stars: present GFMC estimates.

FIG. 5. A comparison at weak couplings g_E between Euclidean MC [25] and ELCE [21] estimates of the string tension, including one-loop correction effects.

FIG. 6. The ratio $M_E/\sqrt{\sigma_E}$ plotted as a function of $\beta = 4/g^2$. Circles: Euclidean MC data [25]; Squares: ELCE data [21].

FIG. 7. Estimates of aM/g^2 , where M is the mass gap, as a function of $\beta = 4/g^2$. Series expansion results from ref. [22]; coupled-cluster estimates by the truncation scheme used by LLewellyn-Smith and Watson [24] and Guo *et al.* [23]; the heavy line is an estimate inferred from Teper [25].

TABLES

TABLE I. Values for the ground-state energy per site $\omega_0(L)$ as a function of lattice size L and coupling λ . Also listed are extrapolated estimates of the bulk limit $L \rightarrow \infty$, together with strong-coupling series estimates [22], and coupled-cluster estimates [24].

$\lambda =$ $L \backslash$	0.5	1.0	2.0	4.0	8.0	16.0
2	-0.04136(2)	-0.16154(5)	-0.5777(1)	-1.753(3)	-4.603(1)	-10.987(2)
3	-0.04108(2)	-0.15795(2)	-0.5593(2)	-1.7092(3)	-4.519(1)	-10.848(2)
4	-0.04108(1)	-0.15800(3)	-0.5582(1)	-1.7021(1)	-4.5048(4)	-10.822(1)
5	-0.04109(1)	-0.1590(2)	-0.5582(1)	-1.7010(2)	-4.501(1)	-10.815(1)
6			-0.5581(1)	-1.7010(4)	-4.500(1)	-10.810(1)
$L \rightarrow \infty$	-0.04109(1)	-0.15795(3)	-0.5581(1)	-1.701(1)	-4.499(1)	-10.805(3)
SC series	-0.04107787	-0.1579360	-0.5581(1)	-1.700(5)		
Coupled cluster	-0.0412(3)	-0.160(2)	-0.566(2)	-1.70(1)		

TABLE II. The mean plaquette value P as a function of lattice size L and coupling λ . Also listed are extrapolated estimates of the bulk limit $L \rightarrow \infty$, together with strong-coupling series estimates [22].

$\lambda =$ $L \backslash$	0.5	1.0	2.0	4.0	8.0	16.0
2	0.1649(5)	0.3116(9)	0.4986(6)	0.6505(10)	0.7548(7)	0.8277(8)
3	0.1623(5)	0.3004(6)	0.4830(8)	0.6382(5)	0.7477(8)	0.8216(5)
4	0.1626(5)	0.3002(5)	0.4803(4)	0.6356(8)	0.7448(10)	0.8197(11)
5	0.1617(3)	0.2991(5)	0.4797(5)	0.6361(7)	0.7459(14)	0.8204(10)
6			0.4816(6)	0.6357(10)	0.7456(12)	0.8214(21)
$L \rightarrow \infty$	0.1622(5)	0.2999(5)	0.480(1)	0.636(1)	0.745(2)	0.820(2)
SC series	0.1620183	0.3001145	0.480918(5)	0.635(5)		

TABLE III. Sample Wilson loop estimates for the 6×6 lattice

λ	$W(1,1)$	$W(2,1)$	$W(2,2)$	$W(3,2)$	$W(3,3)$
0.5	0.1617(3)	0.0305(3)	0.0014(3)	—	—
2.0	0.4816(7)	0.263(2)	0.0932(5)	0.034(1)	0.008(1)
16.0	0.821(2)	0.709(4)	0.569(7)	0.466(8)	0.361(10)













